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General paper

Effective Young's Modulus of a Composite Including Two Groups of Periodically Arranged Inclusions

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Abstract: In this paper, the effects of shape and arrangement of inclusions on the effective Young's modulus of materials are considered by the application of the finite element method through examining a model, which has two groups of periodically arranged inclusions in a matrix. Here, two groups of inclusions A and B are considered, both having equally shaped equally arranged inclusions, which have the same elastic constants but different from the ones of the matrix. This model includes square and hexagonal arrays of inclusions as its special cases. First, the effect of shape of inclusions on the Young's modulus of composite materials is considered from the comparison between the results of rectangular and elliptical inclusions. Next, when the position of group A is fixed, the effect of location of group B is considered. Then, the effective elastic Young's modulus is almost independent of the location of group B if the projected areas of groups A and B are not overlapped. In conclusion, the volume fraction of inclusion and projected area fraction of inclusions are found to be two major parameters controlling the effective Young's modulus of composites.

Key Words: *Micromechanics, Rule of mixture, Composite material, Finite element method, Effective elastic modulus*

1. INTRODUCTION

To predict effective properties of heterogeneous materials from a knowledge of constituents is a classical problem in science and engineering, attracting the attention of a lot of researchers [1-28]. Usually actual composites contain randomly distributed irregular shaped inclusions. However, sometimes square and hexagonal arrays of equally shaped inclusions are treated instead of actual arrangement directly(see Fig. 1); then, some accurate solutions are available for those special cases [23-28], which may be helpful for discussing mechanical properties of composites. Recently, some investigations have been made for disordered array [29] using such as homogenization method [30-37]. These results are useful for evaluating actual composites; however, if the shape and arrangement of inclusions are changed a little, we have to reconsider the effective properties. In other words, there is little discussion about the relation between simple models and random arrangement on the basis of mechanical or physical consideration.

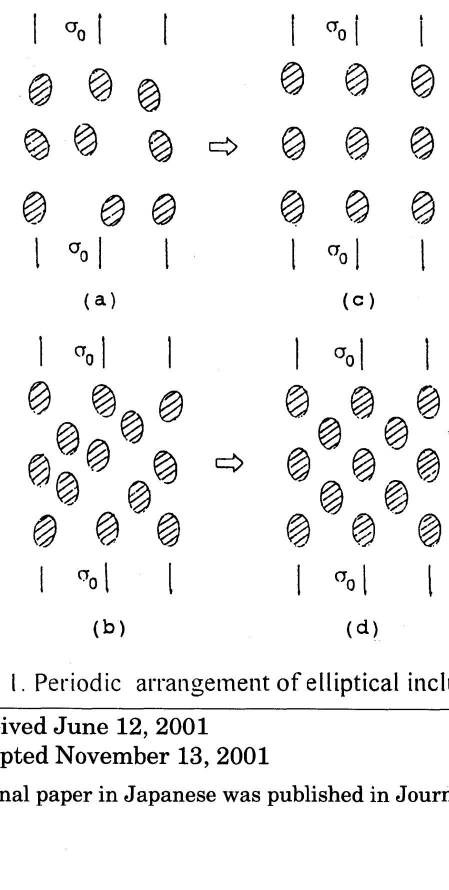

Fig. 1. Periodic arrangement of elliptical inclusions.

In this paper, therefore, the effects of shape and arrangement of inclusions are considered by the application of the finite element method (FEM) through examining a model, which has two groups of periodically arranged inclusions in a matrix (see Fig. 2). Here, two groups of inclusions A and B are considered, both having equally shaped equally arranged inclusions, whose elastic constants are identical but different from the ones of the matrix. This model includes square and hexagonal arrays of inclusions as its special cases. First, the effect of shape of inclusions on the effective Young's modulus will be considered from the comparison between the square arrays of rectangular and elliptical inclusions. Next, the position of group A is fixed; then, the effect of location of group B on the Young's modulus will be considered. Finally, controlling parameters of the effective Young's modulus of composites will be discussed on the basis on those mechanical and physical consideration.

2. METHOD OF ANALYSIS

The two-group-inclusion model shown in Fig. 2 can be analyzed in the following procedure using FEM. Figure 3 (a) shows a unit cell of Fig. 2; then, the boundary conditions can be expressed as follows.

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Here, u and v are displacements in the x and y directions, respectively.

2.1. Boundary Conditions of Fig.3 (a)

(1) Displacements on the lines $x = 0$ and $x = l_x$ have the following relations:

$$u = u_1(y),\quad v = v_1(y) \text{ on the line } x = 0,\ 0 \le y \le l_y \text{ ; then,}$$
$$u = u_1(y) + u_0,\quad v = v_1(y) \text{ on the line } x = l_x,\ 0 \le y \le l_y$$
$$(u_0 : \text{unknown const}). \tag{1}$$

(2) Displacements on the lines $y = 0$ and $y = l_x$ have the following relations:

$$u = u_2(x),\quad v = v_2(x) \text{ on the line } y = 0,\ 0 \le x \le l_x \text{ ; then,}$$
$$u = u_2(x),\quad v = v_2(x) + v_0 \text{ on the line } y = l_y,\ 0 \le x \le l_x$$
$$(v_0 : \text{unknown const}). \tag{2}$$

(3) $$\int_0^{l_x} \sigma_y\big|_{y=0,l_y} dx = \sigma_0 \times l_x,\quad \int_0^{l_y} \sigma_x\big|_{x=0,l_x} dy = 0\,. \tag{3}$$

We have to set the constants u_0, v_0 so as to satisfy Eq. (3). However, since they are still unknown, the following method will be applied instead of solving the given problem directly. First, the following auxiliary problem (1) is solved under the boundary condition as shown in Eqs. (4)-(6). Here, this C_1 is an arbitrary constant (see Fig. 3(b)).

2.2. Boundary Conditions of Fig. 3 (b)

(1)Both displacements on the lines $x = 0$ and $x = l_x$ with $0 \le y \le l_y$ can be expressed as

$$u = u_1(y),\quad v = v_1(y)\,. \tag{4}$$

(2) Displacements on the lines $y = 0$ and $y = l_x$ have the following relations:

$$u = u_2(x),\quad v = v_2(x) \text{ on the line } x = 0,\ 0 \le y \le l_y \text{ ; then,}$$
$$u = u_2(x),\quad v = v_2(x) + C_1 \text{ on the line } x = lx,\ 0 \le y \le l_y$$
$$(C_1 : \text{arbitrary const}). \tag{5}$$

Then, the resultant force F_1 in the x-direction on the boundaries $x = 0$, l_x with $0 \le y \le l_y$ and the resultant force F_2 in y-direction on the boundaries $y = 0$, l_y with $0 \le x \le l_x$ are calculated as shown in Eq.(6).

(3) $$\int_0^{l_y} \sigma_x\big|_{x=0,l_x} dy = F_1,\quad \int_0^{l_x} \sigma_y\big|_{y=0,l_y} dx = F_2\,. \tag{6}$$

Next, the second auxiliary problem (2) is solved under the following boundary conditions (see Fig.3 (c)).

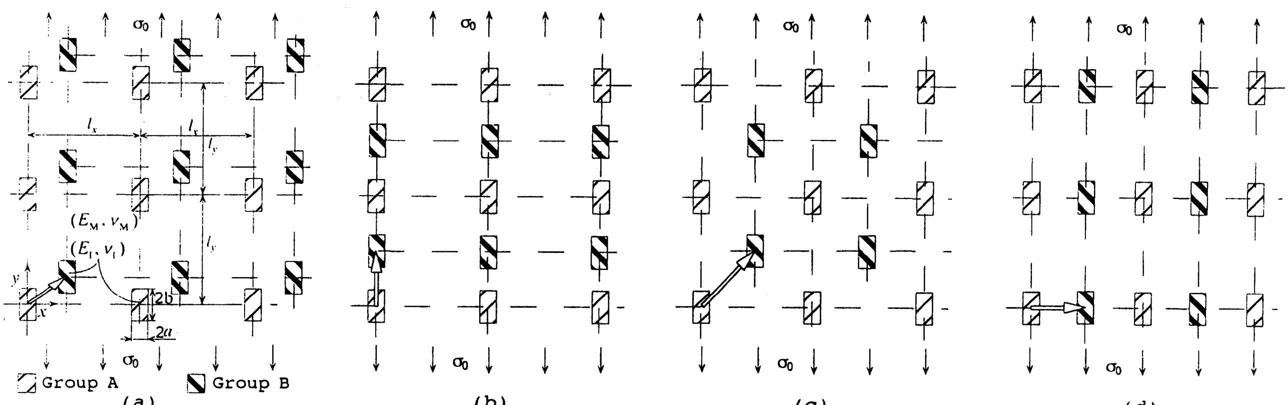

Fig. 2. A model having two groups of periodically arranged inclusions: (a) Model in this study (b) Square array 1 (c) Hexagonal array (d) Square array 2.

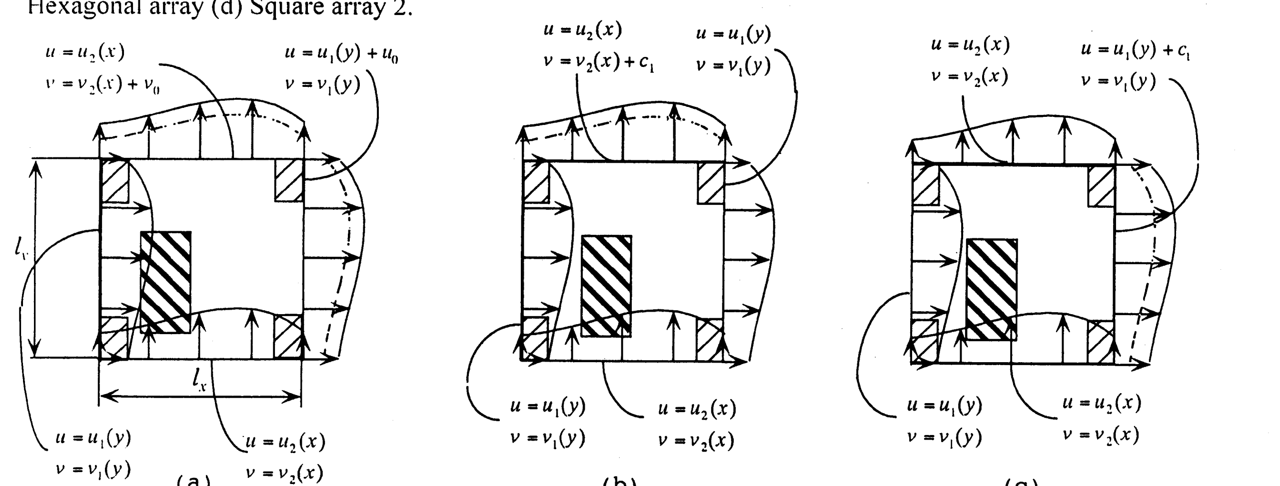

Fig. 3. Boundary conditions and displacement modes: (a) Model in this study (b) Auxiliary problem (1) (c) Auxiliary problem (2).

2.3. Boundary Conditions of Fig. 3 (c)

(1) Displacements on the line $x = 0$ and $x = l_x$ have the following relation:

$$u = u_1(y),\quad v = v_1(y) \text{ on the line } , 0 \le y \le l_y \text{; then,}$$
$$u = u_1(y) + C_1,\quad v = v_1(y) \text{ on the line } x = l_x,\ 0 \le y \le l_y$$
$$(C_1 \text{: arbitrary const}). \qquad (7)$$

(2) Both displacements on the lines $y = 0$ and $y = l_y$ with $0 \le x \le l_x$ can be expressed as:

$$u = u_2(x),\quad v = v_2(x). \qquad (8)$$

Under the conditions (7) and (8), the resultant force F_3 in the x-direction on the boundaries $x = 0$, l_x with $0 \le y \le l_y$ and the resultant force F_4 in the y-direction on the boundaries $y = 0$, l_y with $0 \le x \le l_x$ are calculated as shown in Eq. (9).

(3) $$\int_0^{l_y} \sigma_x \big|_{x=0,l_x} dy = F_3,\quad \int_0^{l_x} \sigma_y \big|_{y=0,l_y} dx = F_4. \qquad (9)$$

The solution for Fig. 3 (a) can be expressed by superposing the solution for Fig. 3 (b) and the solution for Fig. 3 (c) as shown in Eq. (10). Here the solutions of Figs. 3 (a)', (b), (c) denote (σ_a, u_a), (σ_b, u_b), (σ_c, u_c), respectively.

$$\sigma_a = A\sigma_b + B\sigma_c,\quad u_a = Au_b + Bu_c,$$
$$A = \frac{\sigma_0 l_x}{F_2 - F_4(F_1/F_3)},\quad B = -\frac{(F_1/F_3)\sigma_0 l_x}{F_2 - F_4(F_1/F_3)}. \qquad (10)$$

Here, A and B must satisfy

$$A \times F_1 + B \times F_3 = 0$$
$$A \times F_2 + B \times F_4 = \sigma_0 l_x. \qquad (11)$$

The constant displacement u_0, v_0 in Fig. 3(a) can be expressed as the followings:

$$u_0 = BC_1,\quad v_0 = AC_1. \qquad (12)$$

The effective elastic constants of the composite shown in Fig. 3(a) are given by Eq. (13).

$$E_y = \frac{\sigma_0}{(v_0/l_y)} = \frac{\{F_2 - F_4(F_1/F_3)\}/l_x}{C_1/l_y},$$

$$\nu_y = \frac{u_0/l_x}{v_0/l_y} = \frac{F_3 l_y}{F_1 l_x}. \qquad (13)$$

In this paper the effective Young's modulus E $(= E_y)$ is discussed.

3. EFFECT OF SHAPE OF INCLUSION

Plane stress condition with Poisson's ratio $\nu_M = \nu_I = 0.3$ is assumed in the following calculations. Here, (E_M, ν_M) and (E_I, ν_I) are elastic constants of the matrix and inclusion, respectively. First, square array of circular inclusions is analyzed in order to confirm the accuracy of FEM analysis. Then, it is found that Isida-Sato's results [23] coincide with the present results with 1% in most cases.

Next, square arrays of rectangular and elliptical inclusions are compared (see Fig. 4). In Figs. 5 and 6, the results of rectangular inclusions are compared with Murakami's results of elliptical inclusions [24]. When the unit cell has the dimensions $2l_x \times 2l_y$, the volume fractions of inclusion can be expressed as $V_I = ab/(l_x l_y)$ for rectangular inclusion, where a, b are dimensions of rectangle. For elliptical inclusion $V_I = \pi a' b'/(4 l_x l_y)$, where a', b' are radii of ellipse.

In Fig.5, as $V_I \to 0$ with $E_I / E_M = 10^{-5}$ all results coincide with the results of periodically arranged cracks obtained by Isida-Igawa [28]. Similarly, as $V_I \to 0$ with $E_I / E_M = 10^{5}$ all results should coincide with the results of line inclusions although they are not available. The effect of the line inclusion on the Young's modulus seems small because E/E_M is almost unity as $V_I \to 0$.

As shown in Figs 5 and 6, the effective Young's modulus is not equal even though the volume fraction of inclusion is constant. For example, when V_I=0.15 the Young's modulus varies from 1.25 to 2.05 depending on a/l_x, which is the projected area of inclusion. It may be concluded that the effective Young's modulus is identical under the following conditions:

(1) the projected area fractions of inclusions are equal, that is, $a/l_x = a'/l_x$

(2) the volume fractions of inclusions are equal, that is, $ab/(l_x l_y) = \pi a' b'/(4 l_x l_y)$

Figure 4 illustrates the equivalent condition of inclusions.If the condition is satisfied, it may be concluded that the effective Young's modulus is almost equal even though the shape of inclusions differs from rectangle or ellipse. Actual irregularly shaped inclusions, therefore, may be evaluated from equivalent rectangular inclusions with

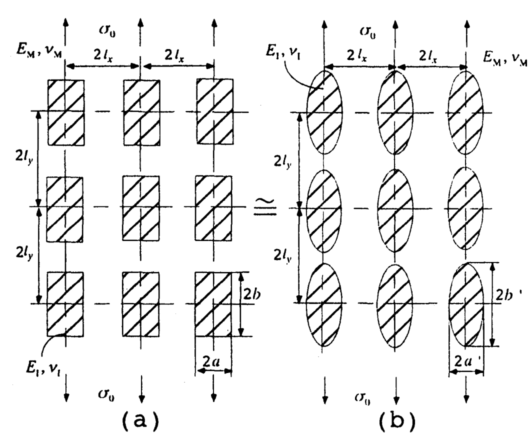

Fig. 4. Elastic modulus is almost equal when (1) $a/l_x = a'/l_x$ and (2) $ab/(l_x l_y) = \pi a' b'/(4 l_x l_y)$.

the application of FEM. This replacement may be effective and efficient if actual inclusions are well-approximated by elliptical or rectangular inclusions.

4. EFFECT OF ARRANGEMENT OF INCLUSION

In this section, the position of group A is fixed in Fig.2. Then, the effect of location of group B on the effective Young's modulus will be considered. When the unit cell has the dimensions $l_x \times l_y$, the volume fractions of inclusion can be expressed as $V_I = 8ab/(l_x l_y)$ for rectangular inclusion, whose dimensions are $2a \times 2b$. Figure 7 shows four types of models (1)-(4) of two groups of periodically arranged inclusions. Here, we set $E_I / E_M = 10^5$ with Poisson's ratio $\nu_M = \nu_I = 0.3$. The central coordinate of group B varies in the range $0 \le x \le l_x/2$, $0 \le y \le l_x/2$. Here, E is the effective Young's modulus in the y-direction (see eqn (13)).

Tables 1-4 indicate relation between E/E_M and central E/E_M vs. central coordinate of group B in model (1). For example, for model (1), $E_I / E_M = 1.157$ for hexagonal array (A), and $E_I / E_M = 1.170$ for square array (B); then, there is no large difference between them. Figures 8-11 indicate E/E_M vs. the central coordinateof group B in models (1)-(4). As shown in these figures, E/E_M takes a maximum value when the central coordinate is on (C), and a minimum value when the central coordinate is on (D). Variation of (E/E_M) normarized by the value of (A) is

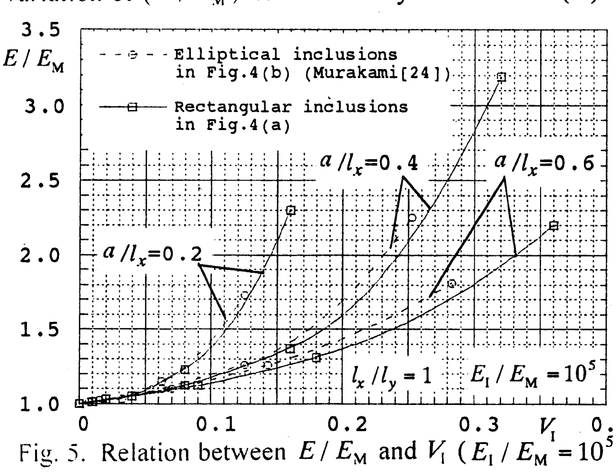

Fig. 5. Relation between E/E_M and V_I ($E_I/E_M = 10^5$).

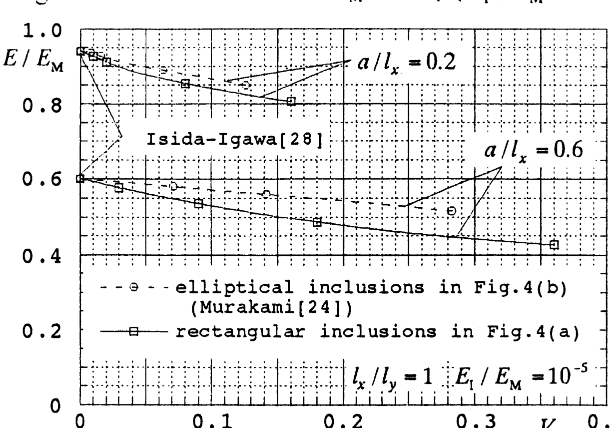

Fig. 6. Relation between E/E_M and V_I ($E_I/E_M = 10^{-5}$).

(3), and 0.90~1.11 for model (4). However, if we compare among the values of (A), (B), (D), the variation becomes small, within about a few percent, that is, 0.99~1.01 for model (1), 0.99~1.03 for model (2), 0.94~1.02 for model (3), and 0.90~1.00 for model (4). It may be concluded that the effective Young's modulus is almost independent of the location of group B if the projected areas of groups A and B are not overlapped. Figure 12 illustrates this condition.

Generally, it may be concluded that the volume fraction of inclusion and projected area fraction of inclusions are two major parameters controlling the effective Young's modulus of composites. In order to evaluate disordered inclusions it may be useful and efficient to analyze an equivalent ordered-rectangular inclusion-model with the application of FEM.

5. CONCLUSIONS

In this paper, the effect of shape and arrangement of inclusions on the effective Young's modulus of heterogeneous materials is considered by the application of FEM through examining a model, which has two groups of periodically arranged inclusions in a matrix.

(1) The effect of shape of inclusions is considered from the results of rectangular and elliptical inclusions. The effective Young's modulus is found to be mainly determined by two major parameters, that is, (i) the area fraction of inclusions projected in tensile direction, and (ii) the volume fraction of inclusion, almost independent of shape of inclusion.
(2) The effect of location of group B is considered when the position of group A is fixed. It may be concluded that the

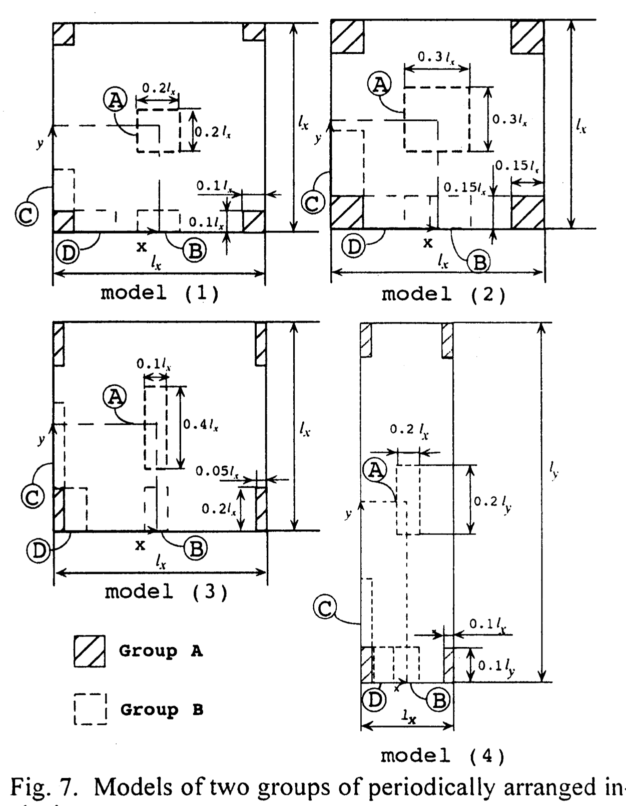

Fig. 7. Models of two groups of periodically arranged inclusions.

Young's modulus is almost independent of the location of group B if the projected areas of groups A and B are not overlapped.

(3) The volume fraction of inclusion and projected area fraction of inclusions are found to be two major parameters controlling the effective Young's modulus of composites. Disordered irregularly-shaped inclusions may be evaluated from equivalent ordered rectangular inclusions with the application of FEM. This replacement may be effective and efficient if actual inclusions are well-approximated by elliptical or rectangular inclusions.

Table 1. E/E_M vs. central coordinate of inclusion B in model(1) ($b/a=1$, $l_x/l_y=1$, $V_I=8ab/(l_x/l_y)=0.08$ $E_I/E_M=10^5$).

y/l_x \ x/l_x	0	0.1	0.2	0.3	0.4	0.5	
0.5	1.194	1.189	1.176	1.164	1.157	1.157	A
0.4	1.200	1.191	1.177	1.165	1.159	1.157	
0.3	1.209	1.198	1.177	1.162	1.159	1.159	
0.2	1.233	1.219	1.200	1.160	1.161	1.162	
0.1	C		1.155	1.161	1.166	1.167	
0		D	1.141	1.163	1.169	1.170	B

Table 2. E/E_M vs. central coordinate of inclusion B in model(2) ($b/a=1$, $l_x/l_y=1$, $V_I=8ab/(l_x/l_y)=0.18$, $E_I/E_M=10^5$).

y/l_x \ x/l_x	0	0.1	0.2	0.3	0.4	0.5	
0.5	1.615	1.586	1.505	1.432	1.388	1.375	A
0.4	1.615	1.574	1.513	1.437	1.387	1.375	
0.3	1.692	1.656	1.590	1.530	1.386	1.379	
0.2				1.426	1.390	1.391	
0.1	C			1.373	1.395	1.411	
0			D	1.354	1.398	1.411	B

Table 3. E/E_M vs. central coordinate of inclusion B in model(3) ($b/a=4$, $l_x/l_y=1$, $V_I=8ab/(l_x/l_y)=0.08$, $E_I/E_M=10^5$).

y/l_x \ x/l_x	0	0.1	0.2	0.3	0.4	0.5	
0.5	1.699	1.538	1.381	1.332	1.316	1.313	A
0.4	1.893	1.772	1.371	1.331	1.318	1.316	
0.3		1.522	1.349	1.328	1.323	1.322	
0.2	C	1.363	1.319	1.325	1.330	1.332	
0.1		1.269	1.299	1.326	1.337	1.340	
0	D	1.233	1.294	1.327	1.339	1.343	B

Table 4. E/E_M vs. central coordinate of inclusion B in model(4) ($b/a=4$, $l_x/l_y=4$, $V_I=8ab/(l_x/l_y)=0.08$, $E_I/E_M=10^5$).

y/l_y \ x/l_x	0	0.1	0.2	0.3	0.4	0.5	
0.50	1.339	1.339	1.339	1.339	1.339	1.339	A
0.40	1.342	1.340	1.340	1.339	1.338	1.338	
0.30	1.353	1.351	1.344	1.336	1.330	1.327	
0.25	1.386	1.377	1.352	1.331	1.317	1.312	
0.20	1.480	1.475	1.463	1.321	1.299	1.293	
0.15			1.364	1.294	1.278	1.274	
0.10	C		1.284	1.260	1.258	1.258	
0		D	1.203	1.226	1.241	1.246	B

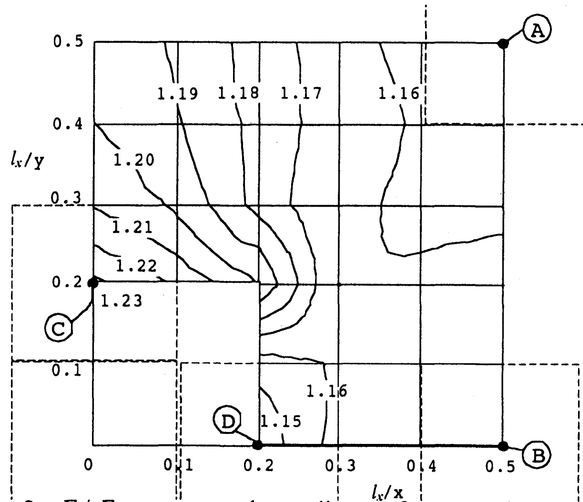

Fig. 8. E/E_M vs. central coordinate of group B in model (1) ($b/a=1$, $l_x/l_y=1$, $V_I=8ab/(l_x/l_y)=0.08$, $E_I/E_M=10^5$).

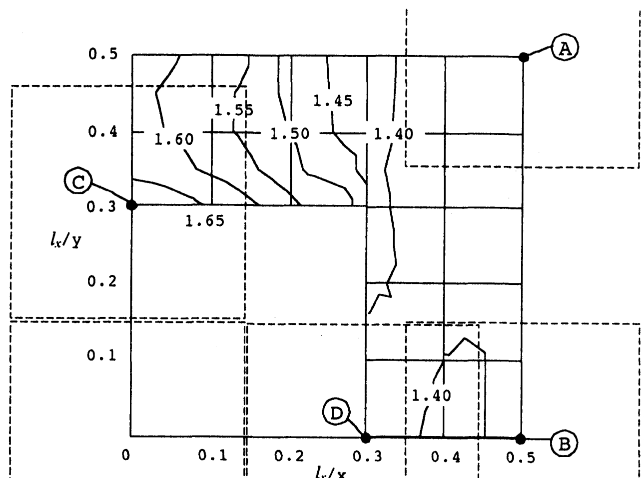

Fig. 9. E/E_M vs. central coordinate of group B in model (2) ($b/a=1$, $l_x/l_y=1$, $V_I=8ab/(l_x/l_y)=0.18$, $E_I/E_M=10^5$).

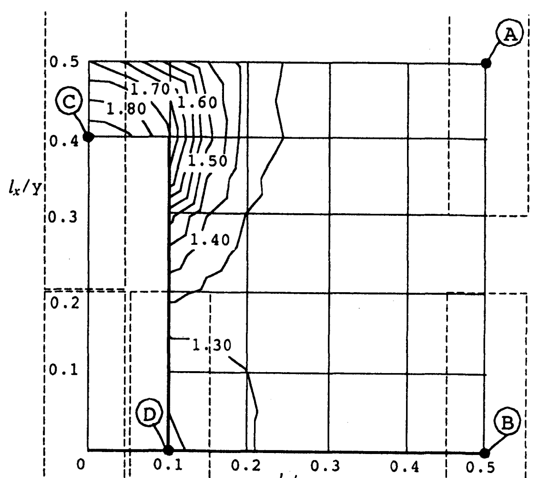

Fig. 10. E/E_M vs. central coordinate of group B in model (3) ($b/a=4$, $l_x/l_y=1$, $V_I=8ab/(l_x/l_y)=0.08$, $E_I/E_M=10^5$).

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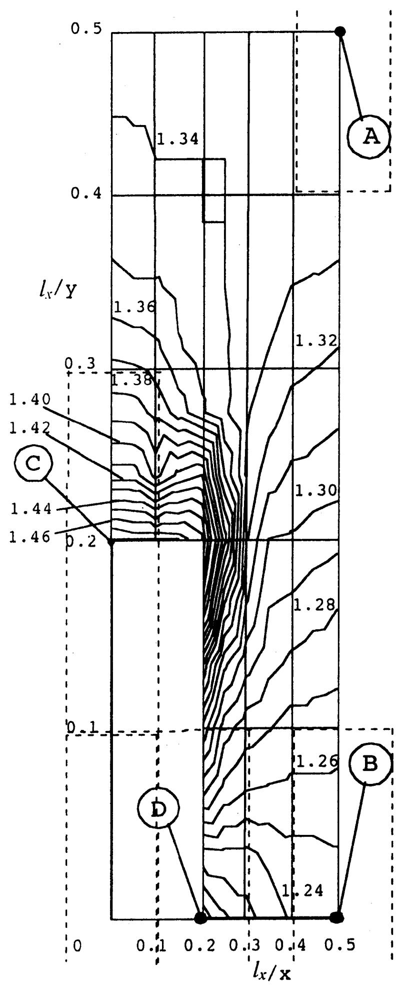

Fig. 11. E/E_M vs. central coordinate of group B in model (4) ($b/a=4$, $l_x/l_y=4$, $V_I=8ab/(l_x/l_y)=0.08$, $E_I/E_M=10^5$).

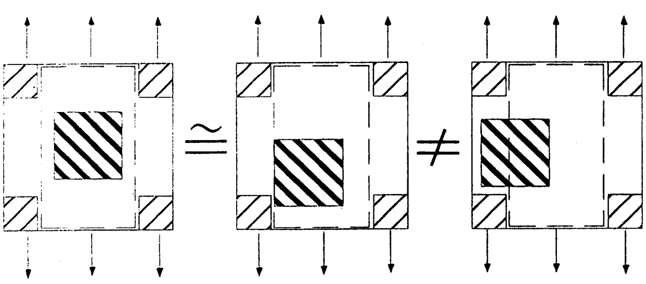

Fig. 12. Effective Young's modulus is almost equal in (a) and (b), but not equal in (c).

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